Path-dependent Hamilton-Jacobi-Bellman equations related to controlled stochastic functional differential systems*

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Abstract: In this paper, a stochastic optimal control problem is investigated in which the system is governed by a stochastic functional differential equation. In the framework of functional Itô calculus, we build the dynamic programming principle and the related Path-dependent Hamilton-Jacobi-Bellman (HJB) equation. We prove that the value function is the viscosity solution of the Path-dependent HJB equation.

Keywords: stochastic functional differential equations; dynamic programming principle; viscosity solution; Path-dependent HJB equations

1 Introduction

In many real world applications, the systems can only be modeled by stochastic systems whose evolutions depend on the past history of the states. So in this paper, we study a stochastic optimal control problem in which the system is described by the following stochastic functional differential equation:

$$X^{\gamma_t,u}(s)=\gamma_t(t)+\int_t^s b(X_r^{\gamma_t,u},u(r))dr+\int_t^s \sigma(X_r^{\gamma_t,u},u(r))dW(r).$$

The cost functional is

$$J(\gamma_t;u(\cdot))=E[\int_t^T L(X_s^{\gamma_t,u},u(s))ds+\Psi(X_T^{\gamma_t,u})].$$

For the initial datum $\gamma_t\in\Lambda$, our optimal control problem is to find an admissible control $u(\cdot)\in\mathcal{U}[t,T]$ so as to minimize the cost functional J. In this case, the value function $V:\Lambda\to\mathbb{R}$ is defined to be

$$V(\gamma_t)=\inf_{u(\cdot)\in\mathcal{U}[t,T]} J(\gamma_t;u(\cdot)).$$

It is well known that dynamic programming and related HJB equations is a powerful approach to solving optimal control problems (see [7], [14] and [11]). Different from the HJB equations derived for stochastic delay systems (see [1], [9] and [10]), we establish the dynamic programming principle and derive the HJB equation in a new framework of functional Itô calculus. The functional Itô calculus and path-dependent PDEs is recently introduced by Dupire [5] (for a recent account of this theory we refer the reader to [2], [3] and [4]). Under the framework of functional Itô calculus, Peng and Wang [13] obtained a nonlinear Feynman-Kac formula for classical solutions of path-dependent PDEs in terms of non-Markovian BSDEs. For the further development, the readers may refer to [12] and [6].

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In our paper, we obtain the following path-dependent HJB equation

$$
\begin{aligned}
&D_tV(\gamma_t)+\inf_{u\in U}G(\gamma_t,u,D_xV,D_{xx}V)=0, \quad \gamma_t\in\Lambda,\\
&V(\gamma_T)=\Psi(\gamma_T), \quad \gamma_T\in\Lambda,
\end{aligned}
$$

where

$$
\begin{aligned}
&G(\gamma_t,u,p,P)=\frac{1}{2}(P\sigma(\gamma_t,u)\sigma(\gamma_t,u)^T)+\langle p,b(\gamma_t)\rangle+L(\gamma_t,u),\\
&\forall(\gamma_t,u,p,P)\in\Lambda\times U\times\mathbb{R}^n\times\mathcal{S}^n.
\end{aligned}
$$

We prove that the value function is the viscosity solution of the Path-dependent HJB equation. In addition, the stochastic verification theorem for the smooth case is also proved.

The paper is organized as follows: in section 2, we present some fundamental results on functional Itô calculus and formulate our stochastic optimal control problem. We establish the dynamic programming principle in section 3. In section 4, we first derive the path-dependent HJB equation and prove the stochastic verification theorem for the smooth case. Then we prove that the value function is the viscosity solution of the path-dependent HJB equation.

2 Problem formulation

2.1 Functional Itô calculus

We adopt the following notations in Dupire [5]. Let $T>0$ be fixed. For each $t\in[0,T]$, we denote by Λ_t the set of càdlàg $\mathbb{R}^d$-valued functions on $[0,t]$.

For each $\gamma\in\Lambda_T$ the value of γ at time $s\in[0,T]$ is denoted by $\gamma(s)$. Thus $\gamma=\gamma(s)_{0\leq s\leq T}$ is a càdlàg function on $[0,T]$ and its value at time s is $\gamma(s)$. The path of γ up to time t is denoted by γ_t, i.e., $\gamma_t=\gamma(s)_{0\leq s\leq t}\in\Lambda_t$. We denote $\Lambda=\bigcup_{t\in[0,T]}\Lambda_t$. For each $\gamma_t\in\Lambda$ and $x\in\mathbb{R}^d$ we denote by $\gamma_t(s)$ the value of γ_t at $s\in[0,t]$ and $\gamma_t^x:=(\gamma_t(s)_{0\leq s<t},\gamma_t(t)+x)$ which is also an element in Λ_t.

We now introduce a distance on Λ. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the inner product and norm in $\mathbb{R}^d$. For each $0\leq t,\bar{t}\leq T$ and $\gamma_t,\bar{\gamma}_{\bar{t}}\in\Lambda$, we denote

$$
\begin{aligned}
&\|\gamma_t\|:=\sup_{s\in[0,t]}|\gamma_t(s)|,\\
&\|\gamma_t-\bar{\gamma}_{\bar{t}}\|:=\sup_{s\in[0,t\vee\bar{t}]}|\gamma_t(s\wedge t)-\bar{\gamma}_{\bar{t}}(s\wedge\bar{t})|,\\
&d_\infty(\gamma_t,\bar{\gamma}_{\bar{t}}):=\sup\nolimits_{0\leq s\leq t\vee\bar{t}}|\gamma_t(s\wedge t)-\bar{\gamma}_{\bar{t}}(s\wedge\bar{t})|+|t-\bar{t}|.
\end{aligned}
$$

It is obvious that Λ_t is a Banach space with respect to $\|\cdot\|$. Since Λ is not a linear space, d_∞ is not a norm.

Definition 2.1. *A functional $u:\Lambda\mapsto\mathbb{R}$ is said to be Λ–continuous at $\gamma_t\in\Lambda$, if for any $\varepsilon>0$ there exists $\delta>0$ such that for each $\bar{\gamma}_{\bar{t}}\in\Lambda$ with $d_\infty(\gamma_t,\bar{\gamma}_{\bar{t}})<\delta$, we have $|u(\gamma_t)-u(\bar{\gamma}_{\bar{t}})|<\varepsilon$.*

u is said to be Λ–continuous if it is Λ–continuous at each $\gamma_t\in\Lambda$.

Definition 2.2. *Let $v:\Lambda\mapsto\mathbb{R}$ and $\gamma_t\in\Lambda$ be given. If there exists $p\in\mathbb{R}^d$, such that*

$$
v(\gamma_t^x)=v(\gamma_t)+\langle p,x\rangle+o(|x|) \ \text{as } x\to0, \ x\in\mathbb{R}^d.
$$

Then we say that v is (vertically) differentiable at γ_t and define $D_x v(\gamma_t) := p$. v is said to be vertically differentiable in Λ if $D_x v(\gamma_t)$ exists for each $\gamma_t \in \Lambda$. We can similarly define the Hessian matrix $D_{xx} v(\gamma_t)$ which is an $\mathbb{S}(d)$-valued functional defined on Λ, where $\mathbb{S}(d)$ is the set of all $d \times d$ symmetric matrices.

For each $\gamma_t \in \Lambda$, we denote

$$\gamma_{t,s}(r) = \gamma_t(r \wedge t), \quad r \in [0, s].$$

It is clear that $\gamma_{t,s} \in \Lambda_s$.

Definition 2.3. *For a given $\gamma_t \in \Lambda$, if we have*

$$v(\gamma_{t,s}) = v(\gamma_t) + a(s-t) + o(|s-t|) \text{ as } s \to t, \ s \geq t,$$

then we say that $v(\gamma_t)$ is (horizontally) differentiable in t at γ_t and define $D_t v(\gamma_t) := a$. u is said to be horizontally differentiable in Λ if $D_t v(\gamma_t)$ exists for each $\gamma_t \in \Lambda$.

Definition 2.4. *Denote by $\mathbb{C}^{j,k}(\Lambda)$ the set of functionals $v := (v(\gamma_t))_{\gamma_t \in \Lambda}$ defined on Λ which are j times horizontally and k times vertically differentiable in Λ such that all these derivatives are Λ–continuous.*

The following Itô's formula was firstly obtained by Dupire [5] and then generalized by Cont and Fournié [2].

Theorem 2.1 (Functional Itô's formula). *Let $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in[0,T]}, P)$ be a probability space. If X is a continuous semi-martingale and u is in $\mathbb{C}^{1,2}(\Lambda)$, then for any $t \in [0, T)$,*

$$\begin{aligned} v(X_t) - v(X_0) = &\int_0^t D_s v(X_s)\, ds + \int_0^t D_x v(X_s)\, dX(s) \\ &+ \frac{1}{2}\int_0^t D_{xx} v(X_s)\, d\langle X\rangle(s) \qquad P-a.s. \end{aligned}$$

2.2 Stochastic optimal control problem

Let W be a d-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}(t)\}_{t\geq 0})$, where $\{\mathcal{F}(t)\}_{t\geq 0}$ is the P-augmentation of the natural filtration generated by the Brownian motion W.

Let $T > 0$ be given. For any $(t, \gamma_t) \in [0, T] \times \Lambda_t$, consider the following controlled stochastic functional differential equation:

$$dX^{\gamma_t,u}(s) = b(X_s^{\gamma_t,u}, u(s))ds + \sigma(X_s^{\gamma_t,u}, u(s))dW(s), \quad s \in [t, T], \tag{2.1}$$

with the initial condition $X_t^{\gamma_t,u} = \gamma_t$, where $u(\cdot) = \{u(s), s \in [t, T]\}$ is a control process taking values in a compact set U of an Euclidean space and $b : \Lambda \times U \to \mathbb{R}^n$, $\sigma : \Lambda \times U \to \mathbb{R}^n \times \mathbb{R}^d$ are given deterministic functions.

The definition of strong solutions of (2.1) and the admissible controls are given as follows.

Definition 2.5. *Given the d-dimensional standard Brownian motion $\{W(s), s \in [0, T]\}$ and the control process $\{u(s), s \in [t, T]\}$, a process $\{X^{\gamma_t,u}(s)\}_{s\in[t,T]}$ is said to be a strong solution of (2.1) on the interval $[t, T]$ and through the initial datum $(t, \gamma_t) \in [0, T] \times \Lambda_t$ if it satisfies the following conditions:*

1. $X_t^{\gamma_t,u} = \gamma_t$;

2. $X^{\gamma_t,u}(s)$ is $\mathcal{F}(s)$-measurable for each $s \in [t, T]$;

3. The process $\{X^{\gamma_t,u}(s), s \in [t, T]\}$ is continuous and it satisfies the following stochastic integral equation $P - a.s.$

$$X^{\gamma_t,u}(s) = \gamma_t(t) + \int_t^s b(X_r^{\gamma_t,u}, u(r))dr + \int_t^s \sigma(X_r^{\gamma_t,u}, u(r))dW(r).$$

Definition 2.6. *For each $t \in [0,T]$, a five-tuples $\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))$ is said to be an admissible control if it satisfies the following conditions:*

1. $(\Omega, \mathcal{F}, P)$ is a complete probability space.

2. $W = \{W(s), s \in [t,T]\}$ is a d-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, P)$ over $[t,T]$ with $W(t) = 0$ a.s. and $\mathcal{F}(s,t) = \sigma\{W(\tau), t \leq \tau \leq s\}$ augmented by all the P-null sets in $\mathcal{F}$.

3. $u : [t,T] \times \Omega \to U$ is an $\mathcal{F}(s,t), s \in [t,T]$-adapted process on $(\Omega, \mathcal{F}, P)$.

4. Under the control process $u(\cdot) = \{u(s), s \in [t,T]\}$, equation (2.1) admits a unique strong solution $X^{\gamma_t,u}(\cdot) = \{X^{\gamma_t,u}(s), s \in [t,T]\}$ defined on $(\Omega, \mathcal{F}, P; \{\mathcal{F}(s,t), s \in [t,T]\})$ and through each initial datum $\gamma_t \in \Lambda_t$.

5. The control process $u(\cdot)$ satisfies

$$E[\int_t^T \mid L(X_s^{\gamma_t,u}, u(s)) \mid ds + \mid \Psi(X_T^{\gamma_t,u}) \mid] < \infty,$$

where $L : \Lambda \times U \to \mathbb{R}$ and $\Psi : \Lambda \to \mathbb{R}$ represent the running and terminal cost functions, respectively.

The collection of admissible controls $\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))$ over the time interval $[t,T]$ is denoted by $\mathcal{U}[t,T]$. We will write $u(\cdot) \in \mathcal{U}[t,T]$ or $\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))$ interchangeably, whenever there is no danger of ambiguity.

Given any initial datum $\gamma_t \in \Lambda$ and any admissible control $u(\cdot) \in \mathcal{U}[t,T]$, we define the cost functional

$$J(\gamma_t; u(\cdot)) = E[\int_t^T L(X_s^{\gamma_t,u}, u(s))ds + \Psi(X_T^{\gamma_t,u})].$$

Then our optimal control problem can be stated as follows:

For each $\gamma_t \in \Lambda$, find a $u(\cdot) \in \mathcal{U}[t,T]$ so as to minimize the cost functional J. In this case, the value function $V : \Lambda \to \mathbb{R}$ is defined to be

$$V(\gamma_t) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(\gamma_t; u(\cdot)). \tag{2.2}$$

We assume b and σ are uniformly continuous and they satisfy the following linear growth and Lispschitz conditions.

Assumption 2.1. *There exists a constant $c > 0$ such that*

$$\mid b(x_t^1, u) - b(x_t^2, u) \mid + \mid \sigma(x_t^1, u) - \sigma(x_t^2, u) \mid \leq c \parallel x_t^1 - x_t^2 \parallel,$$

$\forall (x_t^1, u), (x_t^2, u) \in \Lambda \times U$.

Assumption 2.2. *There exists a constant $K > 0$ such that*

$$\mid b(\Phi_t, u) \mid + \mid \sigma(\Phi_t, u) \mid \leq K(1 + \parallel \Phi_t \parallel), \quad \forall (\Phi_t, u) \in \Lambda \times U.$$

Let L and Ψ be two uniformly continuous real-valued functionals on $\Lambda \times U$ and Λ, respectively.

Assumption 2.3. *There exist a constant $c_1 > 0$ and integer $k_1 > 0$ such that*

$$\mid L(\Phi_t, u) \mid \leq c_1(1 + \parallel \Phi_t \parallel^{k_1}), \quad \mid \Psi(\Phi_T) \mid \leq c_1(1 + \parallel \Phi_T \parallel^{k_1}) \tag{2.3}$$

for all $(\Phi_t, u) \in \Lambda \times U$ and $\Phi_T \in \Lambda$.

Assumption 2.4. *There exists a constant $c > 0$ such that*

$$| L(\Phi_t^1, u) - L(\Phi_t^2, u) | \leq c \parallel \Phi_t^1 - \Phi_t^2 \parallel,$$

$$| \Psi(\Phi_T^1) - \Psi(\Phi_T^2) | \leq c \parallel \Phi_T^1 - \Phi_T^2 \parallel,$$

$\forall (\Phi_T^1, u), (\Phi_T^2, u) \in \Lambda \times U.$

Remark. Let Assumptions 2.1 and 2.2 hold. Then there exists a unique solution X for equation (2.1) (see [8]).

3 Dynamic programming principle

In this section, we build dynamic programming principle for our optimal control problem.

We first give some lemmas.

Denote the set of all continuous functions $\varphi : [0, T] \to \mathbb{R}^n$ by $C([0, T]; \mathbb{R}^n)$.

Lemma 3.1. *(Theorem 2.10 of [14]) Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and (U, d) a Polish space. Let $\xi : [0, T] \times \Omega \to \mathbb{R}^n$ be a continuous process and $\mathcal{F}_t^\xi = \sigma(\xi(s) : 0 \leq s \leq t)$. Then $\varphi : [0, T] \times \Omega \to U$ is $\{\mathcal{F}_t^\xi\}_{t \geq 0}$-adapted if and only if there exists an $\eta \in \mathcal{A}_T^n(U)$ such that $\forall t \in [0, T]$*

$$\varphi(t, \omega) = \eta(t, \xi(\cdot \wedge t, \omega)), \qquad P - a.s. \quad \omega$$

where $\mathcal{A}_T^n(U)$ is the set of all $\{\mathcal{B}_{t+}(C([0, T]; \mathbb{R}^n))\}$-progressively measurable processes.

Lemma 3.2. *Let $\gamma_t \in \Lambda$ and $(\Omega, \mathcal{F}, P, W(\cdot), u(\cdot)) \in \mathcal{U}[t, T]$. Then for any $s \in [t, T]$ and $\mathcal{F}(r, t)_{t \leq r \leq s}$ progressively measurable process $\hat{\gamma}$,*

$$J(\hat{\gamma}_s(\omega); u(\cdot)) = E\{\textstyle\int_s^T L(X_r^{\hat{\gamma}_s, u}, u(r))dr + \Psi(X_T^{\hat{\gamma}_s, u}) \mid \mathcal{F}(s, t)\}(\omega), \qquad P - a.s. \quad \omega. \tag{3.1}$$

Proof. Since $u(\cdot)$ is $\{\mathcal{F}(r, t)\}_{t \leq r \leq T}$-adapted, by Lemma 3.1 there exists a functional $\psi \in \mathcal{A}_T^n(U)$ such that $\forall r \in [t, T]$

$$u(r, \omega) = \psi(r, W(\cdot \wedge r, \omega)), \qquad P - a.s. \quad \omega.$$

For $s \in [t, T]$, let $\hat{\gamma}$ be a $\mathcal{F}(r, t)_{t \leq r \leq s}$ progressively measurable process. We consider the following equation:

$$\begin{aligned} &dX^{\hat{\gamma}_s, \psi}(r) = b(X_r^{\hat{\gamma}_s, \psi}, \psi(r, W(\cdot \wedge r)))dr + \sigma(X_r^{\hat{\gamma}_s, \psi}, \psi(r, W(\cdot \wedge r)))dW(r), \\ &X_s = \hat{\gamma}_s, \quad r \in [s, T], \end{aligned} \tag{3.2}$$

Due to Assumption 2.1 and 2.2, there exists a unique solution of (3.2).

Note that

$$\begin{aligned} & P\{\bar{\omega} : \hat{\gamma}_s(\bar{\omega}) = \hat{\gamma}_s(\omega) \mid \mathcal{F}(s, t)\}(\omega) \\ = \; & E\{I_{\{\hat{\gamma}_s(\bar{\omega}) = \hat{\gamma}_s(\omega)\}} \mid \mathcal{F}(s, t)\}(\omega) \\ = \; & I_{\{\hat{\gamma}_s(\bar{\omega}) = \hat{\gamma}_s(\omega)\}}(\omega) \\ = \; & 1, \quad P - a.s. \quad \omega. \end{aligned}$$

Then there exists an $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that for any fixed $\omega_0 \in \Omega_0$, $\hat{\gamma}_s$ becomes a deterministic function $\hat{\gamma}_s(\omega_0)$ under the new probability space $(\Omega, \mathcal{F}, P(\cdot \mid \mathcal{F}(s,t)(\omega_0))$. On the other hand, for any $r \geq s$,

$$u(r,\omega) = \psi(r, W(\cdot \wedge r)) = \psi(r, \widetilde{W}(\cdot \wedge r, \omega) + W(s,\omega)),$$

where $\widetilde{W}(r) = W(r) - W(s)$ is a standard Brownian motion. Thus, $u(r)$ is adapted to $\widetilde{W}(r)$ for $r \geq s$ under the probability measure $P(\cdot \mid \mathcal{F}(s,t))(\omega_0)$. We obtain that

$$(\Omega, \mathcal{F}, P(\cdot \mid \mathcal{F}(s,t))(\omega_0), \widetilde{W}(\cdot), u(\cdot)) \in \mathcal{U}[t,T]$$

under the new probability space $(\Omega, \mathcal{F}, P(\cdot \mid \mathcal{F}(s,t))(\omega_0))$. By the uniqueness of (3.2), we obtain our result. $\square$

The value function (2.2) has the following properties.

Lemma 3.3. *The value function V is continuous and there exists a constant $c_2 > 0$ and a positive integer k_2 such that, for every $\gamma_t \in \Lambda$,*

$$\mid V(\gamma_t) \mid \leq c_2(1+ \parallel \gamma_t \parallel)^{k_2}.$$

In addition, $\forall \varepsilon > 0$, there exists a constant $\delta > 0$ such that $\mid V(\phi_t) - V(\varphi_s) \mid < \varepsilon$ whenever $\parallel \phi_t - \varphi_s \parallel < \delta$.

Proof. Due to Assumption 2.3, it is easy to prove that V has at most polynomial growth. We only give a brief proof of the continuity property.

Fix $u(\cdot) \in \mathcal{U}[0,T]$ and the initial datum ϕ_t and φ_s, respectively. For $0 \leq t \leq s \leq T$, we have

$$\begin{aligned} & J(\phi_t; u(\cdot)) - J(\varphi_s; u(\cdot)) \\ = \quad & E[\textstyle\int_t^s L(X_r^{\phi_t,u}, u(r))dr + \int_s^T [L(X_r^{\phi_t,u}, u(r)) - L(X_r^{\varphi_s,u}, u(r))]dr + \Psi(X_T^{\phi_t,u}) - \Psi(X_T^{\varphi_s,u})]. \end{aligned}$$

By Assumption 2.3 and 2.4, there exists a constant $c > 0$ such that

$$\mid J(\phi_t; u(\cdot)) - J(\varphi_s; u(\cdot)) \mid \leq c(\mid s - t \mid (1 + E[\parallel X_T^{\phi_t,u} \parallel]) + E[\parallel X_T^{\phi_t,u} - X_T^{\varphi_s,u} \parallel]).$$

Then, by Assumptions 2.1 and 2.2, we can obtain the basic estimations of solution of (2.1) which leads to the following result:

There exists a constant $\delta > 0$ such that

$$\mid J(\phi_t; u(\cdot)) - J(\varphi_s; u(\cdot)) \mid \leq \varepsilon,$$

whenever $\parallel \phi_t - \varphi_s \parallel < \delta$.

Thus, it yields that

$$\mid V(\phi_t) - V(\varphi_s) \mid \leq \varepsilon.$$

This completes the proof. $\square$

The following dynamic programming principle holds.

Theorem 3.1. *Let Assumption 2.1 − 2.4 hold. Then for any $\gamma_t \in \Lambda$ and any $s \in [t,T]$,*

$$V(\gamma_t) = \inf_{u(\cdot) \in \mathcal{U}[t,s]} E[\textstyle\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + V(X_s^{\gamma_t,u})]. \tag{3.3}$$

Proof. Denote the right side of (3.3) by $\bar{V}(\gamma_t)$. For any $\varepsilon > 0$, there exists an $u(\cdot) \in \mathcal{U}[t,T]$ such that

$$\begin{aligned}
& V(\gamma_t) + \varepsilon > J(\gamma_t; u(\cdot)) \\
= \;& E\{\int_t^T L(X_r^{\gamma_t,u}, u(r))dr + \Psi(X_T^{\gamma_t,u})\} \\
= \;& E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + E[\int_s^T L(X_r^{\gamma_t,u}, u(r))dr + \Psi(X_T^{\gamma_t,u}) \mid \mathcal{F}(s,t)]\} \\
= \;& E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + E[\int_s^T L(X_r^{X_s^{\gamma_t,u},u}, u(r))dr + \Psi(X_T^{X_s^{\gamma_t,u},u}) \mid \mathcal{F}(s,t)]\} \\
= \;& E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + J(X_s^{\gamma_t,u})\} \\
\geq \;& E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + V(X_s^{\gamma_t,u})\} \geq \bar{V}(\gamma_t).
\end{aligned}$$

Conversely, by Lemma 3.3, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ such that whenever $\parallel \gamma_t - \hat{\gamma}_t \parallel < \delta$,

$$| J(\gamma_t; u(\cdot)) - J(\hat{\gamma}_t; u(\cdot)) | + | V(\gamma_t) - V(\hat{\gamma}_t) | \leq \varepsilon, \quad \forall u(\cdot) \in \mathcal{U}[t,T].$$

Let $\{D_j\}_{j\geq 1}$ be a Borel partition of $C([0,s];\mathbb{R}^n)$ with diameter $\mathrm{diam}(D_j) < \delta$. Choose $x_s^j \in D_j$. For each j, there is $(\Omega_j, \mathcal{F}_j, P_j, W_j(\cdot), u_j(\cdot)) \in \mathcal{U}[s,T]$ such that

$$J(x_s^j; u_j(\cdot)) \leq V(x_s^j) + \varepsilon.$$

Hence for any $x_s \in D_j$, we can get

$$J(x_s; u_j(\cdot)) \leq J(x_s^j; u_j(\cdot)) + \varepsilon \leq V(x_s^j) + 2\varepsilon \leq V(x_s) + 3\varepsilon. \tag{3.4}$$

By the definition of the admissible control, there is a functional $\psi_j \in \mathcal{A}_T^n(U)$ such that

$$u_j(r,\omega) = \psi_j(r, W_j(\cdot \wedge r, \omega)), \quad P_j - a.s. \quad \omega \in \Omega_j, \quad r \in [s,T].$$

Now for any $(\Omega, \mathcal{F}, P, W(\cdot), u(\cdot)) \in \mathcal{U}[t,T]$, let $X(\cdot) = X^{\gamma_t,u}(\cdot)$ denote the corresponding state trajectory. Define a new control

$$\begin{aligned}
& \tilde{u}(r,\omega) = u(r,\omega), \qquad if \quad r \in [t,s), \\
& \tilde{u}(r,\omega) = \psi_j(r, W(\cdot \wedge r, \omega) - W(s,\omega)), \quad if \quad r \in [s,T], \quad X_r(\omega) \in D_j.
\end{aligned}$$

Clearly, $\tilde{u}(\cdot) \in \mathcal{U}[t,T]$. Thus,

$$\begin{aligned}
V(\gamma_t) \;& \leq J(\gamma_t; \tilde{u}(\cdot)) \\
& = E\{\int_t^T L(X_r^{\gamma_t,\tilde{u}}, \tilde{u}(r))dr + \Psi(X_T^{\gamma_t,\tilde{u}})\} \\
& = E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + E[\int_s^T L(X_r^{\gamma_t,\tilde{u}}, \tilde{u}(r))dr + \Psi(X_T^{\gamma_t,\tilde{u}}) \mid \mathcal{F}(s,t)]\} \\
& = E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + E[\int_s^T L(X_r^{X_s^{\gamma_t,\tilde{u}},\tilde{u}}, \tilde{u}(r))dr + \Psi(X_T^{X_s^{\gamma_t,\tilde{u}},\tilde{u}}) \mid \mathcal{F}(s,t)]\} \\
& = E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + J(X_s^{\gamma_t,\tilde{u}})\} \\
& \leq E\{\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + V(X_s^{\gamma_t,u}) + 3\varepsilon\},
\end{aligned}$$

where the last inequality is due to (3.4). Taking the infimum over $u(\cdot) \in \mathcal{U}[t,T]$, we obtain the result. $\square$

4 Path-dependent HJB equation

4.1 Classical solution

In this subsection, we suppose that the value function V is smooth, i.e., $V \in \mathbb{C}^{1,2}(\Lambda)$.

Theorem 4.1. *Let Assumption* $2.1-2.4$ *hold and the value function* $V \in \mathbb{C}^{1,2}(\Lambda)$*. Then* V *is a solution of the following path-dependent second-order partial differential equation:*

$$\begin{aligned} & D_t V(\gamma_t) + \inf_{u\in U} G(\gamma_t, u, D_x V, D_{xx} V) = 0, \quad \gamma_t \in \Lambda, \\ & V(\gamma_T) = \Psi(\gamma_T), \quad \gamma_T \in \Lambda, \end{aligned} \tag{4.1}$$

where

$$G(\gamma_t, u, p, P) = \frac{1}{2} tr[P\sigma(\gamma_t, u)\sigma(\gamma_t, u)^T] + \langle p, b(\gamma_t, u)\rangle + L(\gamma_t, u),$$

$\forall (\gamma_t, u, p, P) \in \Lambda \times U \times \mathbb{R}^n \times \mathcal{S}^n$ *($\mathcal{S}^n$ is the set of all n order symmetric matrix).*

Proof. Fix $\gamma_t \in \Lambda$ and $u \in U$. Let $X^{\gamma_t,u}(\cdot)$ be the state trajectory corresponding to the initial datum γ_t and the control $u(r) \equiv u$. By Theorem 3.1 and Functional Itô's formula, we obtain

$$\begin{aligned} 0 & \leq \tfrac{E[V(X_s^{\gamma_t,u}) - V(\gamma_t)]}{s-t} + \tfrac{1}{s-t} E[\textstyle\int_t^s L(X_r^{\gamma_t,u}, u) dt] \\ & = \tfrac{1}{s-t} E[\textstyle\int_t^s \{D_t(\gamma_t) + G(X_r^{\gamma_t,u}, u, D_x V(X_r^{\gamma_t,u}), D_{xx} V(X_r^{\gamma_t,u}))\} dt], \end{aligned}$$

Let $s \downarrow t$, we have

$$0 \leq D_t V(\gamma_t) + G(\gamma_t, u, D_x V(\gamma_t), D_{xx} V(\gamma_t)).$$

This result in

$$0 \leq D_t V(\gamma_t) + \inf_{u\in U} G(\gamma_t, u, D_x V(\gamma_t), D_{xx} V(\gamma_t)). \tag{4.2}$$

On the other hand, for any $\varepsilon > 0$, $0 \leq t < s \leq T$ with $s-t > 0$ small enough, there exists a $u_{\varepsilon,s}(\cdot) \in \mathcal{U}[t,T]$ such that

$$V(\gamma_t) + \varepsilon(s-t) \geq E[\int_t^s L(X_r^{\gamma_t,u_{\varepsilon,s}}, u_{\varepsilon,s}(r)) dt + V(X_s^{\gamma_t,u_{\varepsilon,s}})].$$

By Functional Itô's formula,

$$\begin{aligned} \varepsilon & \geq \tfrac{E[V(X_s^{\gamma_t,u_{\varepsilon,s}}) - V(\gamma_t)]}{s-t} + \tfrac{1}{s-t} E[\textstyle\int_t^s L(X_r^{\gamma_t,u_{\varepsilon,s}}, u_{\varepsilon,s}(r)) dt] \\ & = \tfrac{1}{s-t} E[\textstyle\int_t^s \{D_t V(\gamma_t) + G(X_r^{\gamma_t,u_{\varepsilon,s}}, u_{\varepsilon,s}(r), D_x V(X_r^{\gamma_t,u_{\varepsilon,s}}), D_{xx} V(X_r^{\gamma_t,u_{\varepsilon,s}}))\} dt] \\ & \geq \tfrac{1}{s-t} E[\textstyle\int_t^s \{D_t V(\gamma_t) + \inf_{u\in U} G(X_r^{\gamma_t,u_{\varepsilon,s}}, u, D_x V(X_r^{\gamma_t,u_{\varepsilon,s}}), D_{xx} V(X_r^{\gamma_t,u_{\varepsilon,s}}))\} dt], \end{aligned}$$

Let $s \downarrow t$, we have

$$\varepsilon \geq D_t V(\gamma_t) + \inf_{u\in U} G(\gamma_t, u, D_x V(\gamma_t), D_{xx} V(\gamma_t)). \tag{4.3}$$

Combining (4.2) and (4.3), we obtain our conclusion. $\square$

4.1.1 Classical stochastic verification theorem

Now we give the classical stochatic verification theorem.

Theorem 4.2. *Let Assumption* 2.1 − 2.4 *hold.* $V \in \mathbb{C}^{1,2}(\Lambda)$ *is a solution of the HJB equation (4.1). Then*

$$V(\gamma_t) \leq J(\gamma_t; u(\cdot)), \forall u(\cdot) \in \mathcal{U}[t,T], \quad \gamma_t \in \Lambda. \tag{4.4}$$

Futhermore, an admissible pair $(\bar{X}(\cdot), \bar{u}(\cdot))$ *is optimal if and only if*

$$\begin{aligned} D_tV(\bar{X}_s) &= -\inf_{u\in U} G(\bar{X}_s, u, D_xV(\bar{X}_s), D_{xx}V(\bar{X}_s)) \\ &= -G(\bar{X}_s, \bar{u}(s), D_xV(\bar{X}_s), D_{xx}V(\bar{X}_s)), \end{aligned} \tag{4.5}$$

a.e. $s \in [t,T]$, $P-a.s.$

Proof. Applying functional Itô's formula to $V(X_s^{\gamma_t,u})$, we obtain

$$\begin{aligned} V(\gamma_t) = \;& E[\Psi(X_T^{\gamma_t,u})] - E[\textstyle\int_t^T \{D_tV(X_r^{\gamma_t,u}) + \langle D_xV(X_r^{\gamma_t,u}), b(X_r^{\gamma_t,u}, u(r))\rangle \\ & +\tfrac{1}{2}tr(D_{xx}V(X_r^{\gamma_t,u})\sigma(X_r^{\gamma_t,u}, u(r))\sigma(X_r^{\gamma_t,u}, u(r))^T)\}dt] \\ = \;& J(\gamma_t; u(\cdot)) - E[\textstyle\int_t^T L(X_r^{\gamma_t,u}, u(r))dt] - E[\int_t^T \{D_tV(X_r^{\gamma_t,u}) \\ & +\langle D_xV(X_r^{\gamma_t,u}), b(X_r^{\gamma_t,u}, u(r))\rangle + \tfrac{1}{2}tr(D_{xx}V(X_r^{\gamma_t,u})\sigma(X_r^{\gamma_t,u}, u(r))\sigma(X_r^{\gamma_t,u}, u(r))^T)\}dt] \\ \leq \;& J(\gamma_t; u(\cdot)) - E[\textstyle\int_t^T \{D_tV(X_r^{\gamma_t,u}) + \inf_{u\in U} G(X_r^{\gamma_t,u}, u, D_xV(X_r^{\gamma_t,u}), D_{xx}V(X_r^{\gamma_t,u}))\}dt] \\ = \;& J(\gamma_t; u(\cdot)). \end{aligned} \tag{4.6}$$

Then (4.4) follows.

Next, applying (4.6) to $(\bar{X}(\cdot), \bar{u}(\cdot))$, we have

$$V(\gamma_t) = J(\gamma_t; \bar{u}(\cdot)) - E[\int_t^T [D_tV(\bar{X}_s) + G(\bar{X}_s, \bar{u}(s), D_xV(\bar{X}_s), D_{xx}V(\bar{X}_s))]ds].$$

If (4.5) holds, then

$$V(\gamma_t) = J(\gamma_t; \bar{u}(\cdot))$$

and $(\bar{X}(\cdot), \bar{u}(\cdot))$ is optimal.

Conversely, if $(\bar{X}(\cdot), \bar{u}(\cdot))$ is optimal, then

$$V(\gamma_t) = J(\gamma_t; \bar{u}(\cdot)),$$

Applying Itô's formula to $V(\bar{X}_s)$, we have

$$V(\gamma_t) = J(\gamma_t; \bar{u}(\cdot)) - E[\int_t^T [D_tV(\bar{X}_s) + G(\bar{X}_s, \bar{u}(s), D_xV(\bar{X}_s), D_{xx}V(\bar{X}_s))]ds].$$

Note that

$$D_tV(\bar{X}_s) + G(\bar{X}_s, \bar{u}(s), D_xV(\bar{X}_s), D_{xx}V(\bar{X}_s)) \leq 0.$$

We have immediately

$$D_tV(\bar{X}_s) + G(\bar{X}_s, \bar{u}(s), D_xV(\bar{X}_s), D_{xx}V(\bar{X}_s)) = 0.$$

Thus, we get the desired result. □

4.2 Viscosity solution

In this section, under Assumption $2.1-2.4$, we show that the value function V is actually a viscosity solution of the path-dependent HJB equation (4.1). We first give the definition of the viscosity solution of (4.1) as follows.

Definition 4.1. *Let $w \in \mathbb{C}(\Lambda)$, we say that w is a viscosity subsolution of (4.1), if $\forall\delta > 0$, $\forall\Gamma \in \mathbb{C}^{1,2}(\Lambda)$, $\gamma_t \in \Lambda$ satisfying $\Gamma \geq w$ on $\bigcup_{0\leq r\leq\delta} \Lambda_{t+r}$ and $\Gamma(\gamma_t) = w(\gamma_t)$, we have*

$$D_t\Gamma(\gamma_t) + \inf_{u\in U} G(\gamma_t, u, D_x\Gamma, D_{xx}\Gamma) \geq 0.$$

where

$$\begin{aligned}&G(\gamma_t, u, p, P) = \frac{1}{2}tr[P\sigma(\gamma_t, u)\sigma(\gamma_t, u)^T] + \langle p, b(\gamma_t, u)\rangle + L(\gamma_t, u),\\ &\forall(\gamma_t, u, p, P) \in \Lambda \times U \times \mathbb{R}^n \times \mathcal{S}^n.\end{aligned}$$

We say that w is a viscosity supersolution of (4.1), if $\forall\delta > 0$, $\forall\Gamma \in \mathbb{C}^{1,2}(\Lambda)$, $\gamma_t \in \Lambda$ satisfying $\Gamma \leq w$ on $\bigcup_{0\leq r\leq\delta} \Lambda_{t+r}$ and $\Gamma(\gamma_t) = w(\gamma_t)$, we have

$$D_t\Gamma(\gamma_t) + \inf_{u\in U} G(\gamma_t, u, D_x\Gamma, D_{xx}\Gamma) \leq 0.$$

We say that w is a viscosity solution of (4.1) if it is both a viscosity supersolution and a viscosity subsolution of (4.1).

Then we have the following main result:

Theorem 4.3. *Let Assumption $2.1-2.4$ hold. Then the value function V is a viscosity solution of the path-dependent equation (4.1).*

Proof. (1) For any $\delta > 0$ and $\Gamma \in \mathbb{C}^{1,2}(\Lambda)$, let $\Gamma \geq V$ on $\bigcup_{0\leq r\leq\delta} \Lambda_{t+r}$ and $\Gamma(\gamma_t) = V(\gamma_t)$. Fix a $u \in U$. Let $X^{\gamma_t,u}(\cdot)$ be the solution of (2.1) with the control $u(s) \equiv u$. Then by Functional Itô's formula, for $t \leq s \leq t+\delta$, we have

$$\begin{aligned}&E[\Gamma(X_s^{\gamma_t,u})] - \Gamma(\gamma_t)]\\ =\;& E[\textstyle\int_t^s (D_t\Gamma(X_r^{\gamma_t,u}) + G(\gamma_t, u, D_x\Gamma(X_r^{\gamma_t,u}), D_{xx}\Gamma(X_r^{\gamma_t,u})) - L(X_r^{\gamma_t,u}, u))dr].\end{aligned} \tag{4.7}$$

On the other hand, for any $s \in [t, t+\delta]$, by the dynamic programming principle, we have

$$V(\gamma_t) \leq E[\int_t^s L(X_r^{\gamma_t,u}, u)dr + V(X_s^{\gamma_t,u})].$$

Notice that $\Gamma \geq V$ and $\Gamma(\gamma_t) = V(\gamma_t)$. By (4.7), we get

$$\begin{aligned}0 \leq\;& E[\textstyle\int_t^s L(X_r^{\gamma_t,u}, u)dr + V(X_s^{\gamma_t,u})] - V(\gamma_t)\\ \leq\;& E[\textstyle\int_t^s L(X_r^{\gamma_t,u}, u)dr + \Gamma(X_s^{\gamma_t,u})] - \Gamma(\gamma_t)\\ \leq\;& E[\textstyle\int_t^s D_t\Gamma(X_r^{\gamma_t,u}) + G(\gamma_t, u, D_x\Gamma(X_r^{\gamma_t,u}), D_{xx}\Gamma(X_r^{\gamma_t,u}))dr],\end{aligned}$$

Dividing both sides by $(s-t)$, we have

$$0 \leq \tfrac{1}{s-t} E[\textstyle\int_t^s D_t\Gamma(X_r^{\gamma_t,u}) + G(\gamma_t, u, D_x\Gamma(X_r^{\gamma_t,u}), D_{xx}\Gamma(X_r^{\gamma_t,u}))dr].$$

Now let $s \downarrow t$. This leads to

$$D_t\Gamma(\gamma_t) + \inf_{u\in U} G(\gamma_t, u, D_x\Gamma(\gamma_t), D_{xx}\Gamma(\gamma_t)) \geq 0.$$

(2) Next, we want to prove V is a viscosity supersolution.

For any $\delta > 0$ and $\Gamma \in \mathbb{C}^{1,2}(\Lambda)$, let $\Gamma \leq V$ on $\bigcup_{0\leq r\leq\delta} \Lambda_{t+r}$ and $\Gamma(\gamma_t) = V(\gamma_t)$. For any $\varepsilon > 0$ and $\delta > s > t$ (with $s - t > 0$ small enough), by dynamic programming principle, we can find a $u(\cdot) = u_{\varepsilon,s}(\cdot) \in \mathcal{U}[t, T]$ such that

$$\begin{aligned} 0 \quad & \geq E[V(\gamma_t) - \Gamma(\gamma_t) - V(X_s^{\gamma_t,u}) + \Gamma(X_s^{\gamma_t,u})] \\ & \geq -\varepsilon(s-t) + E[\textstyle\int_t^s L(X_r^{\gamma_t,u}, u(r))dr + \Gamma(X_s^{\gamma_t,u}) - \Gamma(\gamma_t)]. \end{aligned} \tag{4.8}$$

Then applying Functional Itô's formula to $\Gamma(X_s^{\gamma_t,u})$, we have

$$\begin{aligned} & E[\Gamma(X_s^{\gamma_t,u})] - \Gamma(\gamma_t) \\ = \quad & E[\textstyle\int_t^s (D_t\Gamma(X_r^{\gamma_t,u}) + G(\gamma_t, u, D_x\Gamma(X_r^{\gamma_t,u}), D_{xx}\Gamma(X_r^{\gamma_t,u})) - L(X_r^{\gamma_t,u}, u))dr]. \end{aligned} \tag{4.9}$$

Dividing both sides of (4.8) by $(s - t)$, we get

$$\begin{aligned} \varepsilon \geq \quad & \tfrac{1}{s-t} E[\textstyle\int_t^s L(X_r^{\gamma_t,u}, u(r))dr \\ & + \textstyle\int_t^s (D_t\Gamma(X_r^{\gamma_t,u}) + G(\gamma_t, u, D_x\Gamma(X_r^{\gamma_t,u}), D_{xx}\Gamma(X_r^{\gamma_t,u})) - L(X_r^{\gamma_t,u}, u))dr] \\ \geq \quad & \tfrac{1}{s-t} E[\textstyle\int_t^s (D_t\Gamma(X_r^{\gamma_t,u}) + \inf_{u\in U} G(\gamma_t, u, D_x\Gamma(X_r^{\gamma_t,u}), D_{xx}\Gamma(X_r^{\gamma_t,u}))dr]. \end{aligned}$$

Let $s \downarrow t$. It yields

$$D_t\Gamma(\gamma_t) + \inf_{u\in U} G(\gamma_t, u, D_x\Gamma(\gamma_t), D_{xx}\Gamma(\gamma_t)) \leq 0.$$

Thus, $V(\gamma_t)$ is a viscosity supersolution. This completes the proof. $\square$

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